Seminar on Knot Theory

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For the first part of the seminar we will introduce the key objects of classical knot theory, namely the Alexander module, the Alexander polynomial and Seifert matrices. In the first 6 talks we mostly follow [Li], but the speakers should also look at the other references [BZH, Ro] to get a broader knowledge. Googling and looking at wikipedia for more results, ideas and examples is warmly encouraged.

The goal is not to cover the material in precisely 90 minutes, instead we want to have ample time for discussions and side excursions.

List of the topics

1. **Seifert matrices and the Alexander module** (Andreas Wicher)
2. **The Alexander polynomial (I)** (Julian Hannes)
3. **The Alexander polynomial (II)** (Gerrit Herrmann)
4. **S-equivalence and the Conway polynomial** (Jakob Schubert)
5. **Levine-Tristram signatures and sliceness (I)** (Stefan Wolf)
6. **Levine-Tristram signatures and sliceness (II)** (Stefan Wolf)
7. **Morse Theory** (Daniel Grünbaum)
8. **Alexander Theorem** (Alexander Neumann)
9. **Dehn's Lemma, the Loop Theorem and the Sphere Theorem**
10. **Stallings' fibering theorem** (Bruno Mazorra)

Detailed description

1. **Seifert matrices and the Alexander module** ([Li, pp. 49, 51-55, 69-70]). Introduce the concept of a presentation of a module. Define the Seifert form associated to the Seifert surface of a link, and compute it on at least one non-trivial example. Construct the infinite cyclic cover of a link exterior by cut-and-paste along a Seifert surface. Show that its first homology group is a module over $\mathbb{Z}[t, t^{-1}]$ (the *Alexander module*), which can be presented by means of the Seifert form.

2. **The Alexander polynomial (I)** ([Li, pp. 50-51, 55-60]). State without proof the algebraic relation between two different presentations of a module, and define the elementary ideals. Define the Alexander polynomial of a link as the generator of the first elementary ideal of the Alexander module. Describe the Alexander polynomial in terms of a Seifert matrix. Prove the main elementary properties and compute some examples.

3. **The Alexander polynomial (II)** ([Li, pp. 10, 60-64]). Describe the satellite construction. Give a formula for the Alexander polynomial of a satellite knot in terms of the Alexander polynomial of its companion and pattern, and discuss some consequences. Give a description of the Alexander polynomial as the characteristic polynomial of the translation map.

4. **S-equivalence of Seifert matrices** ([Li, pp. 79-83]). Show that any two Seifert surfaces of a given link are related by surgery transformations. Define S-equivalence of square matrices, and show that any two Seifert matrices for the same link are S-equivalent. If time permits, introduce and discuss the Conway-normalization of the Alexander polynomial.

5. **Levine-Tristram signatures and sliceness (I)** ([Li, pp. 84-86] + extra reference to be found). Introduce the ω-signature (also called the *Levine-Tristram* signature) of a link, and show that it is well defined. Discuss its behaviour under reflection. Compute the Levine-Tristram signature for the trefoil knot, and relate the calculation to the previous result. Define the concept of slice and ribbon knots. Show that $K\sharp\overline{rK}$ is slice. Show (at least pictorially) that ribbon implies slice, and state the slice-ribbon conjecture.

6. **Levine-Tristram signatures and sliceness (II)** ([Li, pp. 87-91]). Prove or at least sketch the proof of the technical results leading to the description of the Seifert matrix for a slice knot. Show that the Alexander polynomial of a slice knot can be written in a very special form, and that the Levine-Tristram signature is 0 outside of the roots of the Alexander polynomial.

7. **Morse Theory** [Mi, Ni] Introduce Morse functions and handle decompositions. Show that Morse functions on compact manifolds lead to handle decompositions and CW-structures.

8. **Alexander Theorem** Use Morse theory to show that every sphere and torus in S^3 cobound a ball respectively solid torus.

9. **Dehn's Lemma, the Loop Theorem and the Sphere Theorem** ([Ro, Appendix B][Ha, Chapter 3]) State and discuss the relevance of these three theorems. Provide a proof for Dehn's Lemma.

10. **Stallings' fibering theorem** (Chapter 5 and Proposition 8.33 of [BZH]) Discuss fundamental groups of fibered knots and 3-manifolds. Show that the Alexander polynomial of a fibered knot is monic. State and prove Stallings' fibering theorem for knots.

References

[AFW] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, EMS Series of Lectures in Mathematics (2015)

[BZH] G. Burde, H. Zieschang and M. Heusener, *Knots*, 3rd fully revised and extended edition. De Gruyter Studies in Mathematics 5 (2014).

[Ha] A. Hatcher. *Notes on basic 3-manifold topology* `http://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf`

[Li] W.B.R. Lickorish, *An introduction to Knot Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1997

[Mi] J. Milnor. *Morse Theory*, Annals of Mathematics Studies. No. 51. Princeton, N.J.: Princeton University Press (1963).

[Ni] L. Nicolaescu. *An invitation to Morse theory*, Universitext, Springer Verlag (2011)

[Ro] D. Rolfsen. *Knots and links*, Mathematics Lecture Series. 7. Houston, TX: Publish or Perish. (1990).